

FORMULAIRE RELATIF AUX OPÉRATEURS

Soient U et V deux champs scalaires et $\vec{a}$ et $\vec{b}$ deux champs vectoriels.

1. Formules portant sur un seul champ:

1. $\vec{\nabla} \cdot (\vec{\nabla} U) = \vec{\nabla}^2 U$	soit $\vec{\nabla} \cdot (\vec{\nabla} U) = \Delta U$
2. $\vec{\nabla} \wedge (\vec{\nabla} U) = \vec{0}$	soit $\vec{\nabla} \wedge (\vec{\nabla} U) = \vec{0}$
3. $\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{a}) = \vec{0}$	soit $\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{a}) = \vec{0}$
4. $\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \vec{\nabla}^2 \vec{a}$	soit $\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \vec{\nabla}^2 \vec{a}$

2. Formules portant sur deux champs:

5. $\vec{\nabla}(UV) = U \vec{\nabla}(V) + V \vec{\nabla}(U)$	soit $\vec{\nabla}(UV) = U \vec{\nabla}(V) + V \vec{\nabla}(U)$
6. $\vec{\nabla} \cdot (U \vec{a}) = \vec{\nabla} U \cdot \vec{a} + U \vec{\nabla} \cdot \vec{a}$	soit $\vec{\nabla} \cdot (U \vec{a}) = \vec{\nabla} U \cdot \vec{a} + U \vec{\nabla} \cdot \vec{a}$
7. $\vec{\nabla} \wedge (U \vec{a}) = (\vec{\nabla} U) \wedge \vec{a} + U (\vec{\nabla} \wedge \vec{a})$	soit $\vec{\nabla} \wedge (U \vec{a}) = (\vec{\nabla} U) \wedge \vec{a} + U (\vec{\nabla} \wedge \vec{a})$
8. $\vec{\nabla} \cdot (\vec{a} \wedge \vec{b}) = \vec{b} \cdot (\vec{\nabla} \wedge \vec{a}) - \vec{a} \cdot (\vec{\nabla} \wedge \vec{b})$	soit $\vec{\nabla} \cdot (\vec{a} \wedge \vec{b}) = \vec{b} \cdot (\vec{\nabla} \wedge \vec{a}) - \vec{a} \cdot (\vec{\nabla} \wedge \vec{b})$
9. $\vec{\nabla} \wedge (\vec{a} \wedge \vec{b}) = (\vec{\nabla} \cdot \vec{b}) \vec{a} - (\vec{\nabla} \cdot \vec{a}) \vec{b} + (\vec{b} \cdot \vec{\nabla}) \vec{a} - (\vec{a} \cdot \vec{\nabla}) \vec{b}$	
soit $\vec{\nabla} \wedge (\vec{a} \wedge \vec{b}) = (\vec{\nabla} \cdot \vec{b}) \vec{a} - (\vec{\nabla} \cdot \vec{a}) \vec{b} + (\vec{b} \cdot \vec{\nabla}) \vec{a} - (\vec{a} \cdot \vec{\nabla}) \vec{b}$	
10. $\vec{\nabla} \cdot (\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\vec{\nabla} \wedge \vec{b}) + \vec{b} \cdot (\vec{\nabla} \wedge \vec{a}) + (\vec{b} \cdot \vec{\nabla}) \vec{a} + (\vec{a} \cdot \vec{\nabla}) \vec{b}$	
soit $\vec{\nabla} \cdot (\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\vec{\nabla} \wedge \vec{b}) + \vec{b} \cdot (\vec{\nabla} \wedge \vec{a}) + (\vec{b} \cdot \vec{\nabla}) \vec{a} + (\vec{a} \cdot \vec{\nabla}) \vec{b}$	

3. Expressions des opérateurs dans divers systèmes de coordonnées:

a. Gradient:

* cartésiennes: $\vec{\nabla} U = \left(\frac{\partial U}{\partial x} \right) \vec{e}_x + \left(\frac{\partial U}{\partial y} \right) \vec{e}_y + \left(\frac{\partial U}{\partial z} \right) \vec{e}_z$

* cylindriques: $\vec{\nabla} U = \left(\frac{\partial U}{\partial r} \right) \vec{e}_r + \frac{1}{r} \left(\frac{\partial U}{\partial \theta} \right) \vec{e}_\theta + \left(\frac{\partial U}{\partial z} \right) \vec{e}_z$

* sphériques: $\vec{\nabla} U = \left(\frac{\partial U}{\partial r} \right) \vec{e}_r + \frac{1}{r} \left(\frac{\partial U}{\partial \theta} \right) \vec{e}_\theta + \frac{1}{r \sin \theta} \left(\frac{\partial U}{\partial \varphi} \right) \vec{e}_\varphi$

b. Divergence:

* cartésiennes: $\vec{\nabla} \cdot \vec{a} = \left(\frac{\partial a_x}{\partial x} \right) + \left(\frac{\partial a_y}{\partial y} \right) + \left(\frac{\partial a_z}{\partial z} \right)$

* cylindriques: $\vec{\nabla} \cdot \vec{a} = \frac{1}{r} \left(\frac{\partial r a_r}{\partial r} \right) + \frac{1}{r} \left(\frac{\partial a_\theta}{\partial \theta} \right) + \left(\frac{\partial a_z}{\partial z} \right)$

* sphériques: $\vec{\nabla} \cdot \vec{a} = \frac{1}{r^2} \left(\frac{\partial r^2 a_r}{\partial r} \right) + \frac{1}{r \sin \theta} \left(\frac{\partial a_\theta \sin \theta}{\partial \theta} \right) + \frac{1}{r \sin \theta} \left(\frac{\partial a_\varphi}{\partial \varphi} \right)$

c. Rotationnel:

* cartésiennes: $\vec{\text{rot}} \vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) \vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) \vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \vec{e}_z$

* cylindriques: $\vec{\text{rot}} \vec{a} = \left(\frac{1}{r} \frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z} \right) \vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \right) \vec{e}_\theta + \frac{1}{r} \left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta} \right) \vec{e}_z$

* sphériques:

$$\vec{\text{rot}} \vec{a} = \frac{1}{r \sin \theta} \left(\frac{\partial (a_\phi \sin \theta)}{\partial \theta} - \frac{\partial a_\theta}{\partial \phi} \right) \vec{e}_r + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial a_r}{\partial \phi} - \frac{\partial (r a_\phi)}{\partial r} \right) \vec{e}_\theta + \frac{1}{r} \left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta} \right) \vec{e}_\phi$$

d. Laplacien:

* cartésiennes: $\Delta U = \left(\frac{\partial^2 U}{\partial x^2} \right) + \left(\frac{\partial^2 U}{\partial y^2} \right) + \left(\frac{\partial^2 U}{\partial z^2} \right)$

* cylindriques: $\Delta U = \left(\frac{\partial^2 U}{\partial r^2} \right) + \frac{1}{r} \left(\frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \left(\frac{\partial^2 U}{\partial \theta^2} \right) + \left(\frac{\partial^2 U}{\partial z^2} \right)$

* sphériques: $\Delta U = \left(\frac{\partial^2 U}{\partial r^2} \right) + \frac{2}{r} \left(\frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial^2 U}{\partial \phi^2} \right) + \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) \right)$

4. Action des opérateurs sur le trièdre de base:

a. coordonnées cylindriques:

$$\vec{\text{rot}} \vec{e}_r = \vec{0}; \quad \vec{\text{rot}} \vec{e}_\theta = \frac{\vec{e}_z}{r}; \quad \vec{\text{rot}} \vec{e}_z = \vec{0} \quad \text{pour le rotationnel}$$

$$\text{div} \vec{e}_r = \frac{1}{r}; \quad \text{div} \vec{e}_\theta = 0; \quad \text{div} \vec{e}_z = 0 \quad \text{pour la divergence}$$

b. coordonnées sphériques:

$$\vec{\text{rot}} \vec{e}_r = \vec{0}; \quad \vec{\text{rot}} \vec{e}_\theta = \frac{\vec{e}_\phi}{r}; \quad \vec{\text{rot}} \vec{e}_\phi = \frac{\cos \theta}{r \sin \theta} \vec{e}_r - \frac{\vec{e}_\theta}{r} \quad \text{pour le rotationnel}$$

$$\text{div} \vec{e}_r = \frac{2}{r}; \quad \text{div} \vec{e}_\theta = \frac{1}{r \tan \theta}; \quad \text{div} \vec{e}_\phi = 0 \quad \text{pour la divergence}$$